# A QUASI-PAUCITY PROBLEM

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**1. Introduction.** A cartographer of the diophantine landscape is compelled to acknowledge the distinguished position occupied by the investigation of diophantine systems in which there are believed to be few other than the obvious solutions. When these systems are symmetric, the task of verifying such a belief has come to be called a *paucity problem.* Although the literature surrounding this topic is by now extensive when the underlying summands are perfect powers (see, for example, the sources recorded in the bibliography), little is known for more general situations. The object of this note is to establish that the number of solutions of certain systems of additive equations is dominated, in essence, by the diagonal contribution alone.

In order to state our main conclusion precisely, we require some notation. Suppose that $t$ is a positive integer, and let $f_1(x),\dots,f_t(x)$ be polynomials with rational coefficients of respective degrees $k_1,\dots,k_t$. When $P$ is a positive number, denote by $S_s(P;\mathbf{f})$ the number of integral solutions of the simultaneous equations

$$\sum_{i=1}^{s}(f_j(x_i)-f_j(y_i))=0 \quad (1\leqslant j\leqslant t), \tag{1}$$

with $1\leqslant x_i,y_i\leqslant P$ $(1\leqslant i\leqslant s)$.

**Theorem 1.** *Suppose that the polynomials $f_i(x)\in\mathbb{Q}[x]$ $(1\leqslant i\leqslant t)$ satisfy the condition that $1,f_1,\dots,f_t$ are linearly independent over $\mathbb{Q}$. Suppose also that $A$ is a positive number sufficiently large in terms of $t$, $\mathbf{k}$ and the coefficients of $f_1,\dots,f_t$. Then whenever* $\max\{k_1,\dots,k_t\}\geqslant 2$ *and* $P\geqslant 3$, *one has*

$$S_{t+1}(P;f_1,\dots,f_t)\ll P^{t+1}(\log P)^A.$$

Plainly, those solutions of the system (1) in which $x_1,\dots,x_s$ are simply a permutation of $y_1,\dots,y_s$ provide a contribution to $S_{t+1}(P;\mathbf{f})$ that ensures the lower bound

$$S_{t+1}(P;\mathbf{f})\geqslant (t+1)!P^{t+1}+O_t(P^t). \tag{2}$$

Thus we may assert that the conclusion of the theorem is somewhat close to a paucity result. We remark that the bound recorded in the theorem was already available from work of Wooley [28] in the special case wherein $f_i(x)=x^{k_i}$ $(1\leqslant i\leqslant t)$ and $1\leqslant k_1<k_2<\dots<k_t$. Moreover, when $t=1$ and $f_1(x)$ is a cubic polynomial, it follows from Theorem 2 of Wooley [32] that

$$S_2(P;f_1)=2P^2+O_\varepsilon(P^{5/3+\varepsilon}).$$

Aside from the inherent interest of paucity problems, estimates of the type presented in the above theorem have potential for application in the sharpest versions, due to Parsell [19], of the new iterative methods of Vaughan and Wooley (see especially [23], [24], [27], [31]) involving exponential sums over smooth numbers.

**2. Preliminary skirmishing.** Before advancing to the argument described in the next section, we prepare an eliminant polynomial and discuss some associated properties. We refer to an ordered $t$-tuple $(f_1,\dots,f_t)$ of polynomials with rational coefficients as being *well-conditioned* when

(1) one has $f_i(x)\in\mathbb{Z}[x]$ $(1\leqslant i\leqslant t)$, and
(2) one has $f_i(0)=0$ $(1\leqslant i\leqslant t)$, and
(3) the degrees $k_i$ of the polynomials $f_i(x)$ satisfy $1\leqslant k_1<k_2<\dots<k_t$.

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By substituting the polynomial $f_i(x) - f_i(0)$ in place of $f_i(x)$ $(1 \leqslant i \leqslant t)$ in the system (1), one may plainly suppose that $f_i(0) = 0$ $(1 \leqslant i \leqslant t)$. Thus, on replacing the equations (1) by suitable linear combinations thereof, it is apparent that whenever $f_i(x) \in \mathbb{Q}[x]$ $(1 \leqslant i \leqslant t)$ satisfy the condition that $1, f_1, \ldots, f_t$ are linearly independent over $\mathbb{Q}$, then there is no loss of generality in supposing instead that $\mathbf{f}$ is a well-conditioned $t$-tuple. Moreover, the coefficients of the polynomials in the new system plainly depend at most on those in the original system.

It is convenient in what follows to refer to a polynomial $F(\mathbf{x}) \in \mathbb{Z}[x_1, \ldots, x_t]$ as being *asymptotically definite* if there exists a number $C$ with the property that whenever $x_i > C$ $(1 \leqslant i \leqslant t)$, then one has

$$|F(x_1, \ldots, x_t)| \geqslant 1.$$

Finally, we write $f'(x)$ for the derivative of the polynomial $f(x)$. As a prerequisite to a discussion of eliminant polynomials, we introduce a generalisation of the Vandermonde determinant

$$V_t(\mathbf{x}) = \prod_{1 \leqslant i < j \leqslant t} (x_j - x_i) \;=\; \det(x_j^{i-1})_{1 \leqslant i,j \leqslant t}.$$

**Lemma 1.** *Suppose that $(f_1, \ldots, f_t)$ is a well-conditioned $t$-tuple of polynomials with respective degrees $k_1, \ldots, k_t$. Then there exists an asymptotically definite polynomial $\Theta = \Theta(\mathbf{x}; \mathbf{f})$ with the property that*

$$\det(f_i'(x_j))_{1 \leqslant i,j \leqslant t} = V_t(\mathbf{x})\Theta(\mathbf{x}; \mathbf{f}). \tag{3}$$

*Moreover, the total degree of $\Theta$ is*

$$d = \sum_{i=1}^{t} k_i - \frac{t(t+1)}{2}.$$

*Proof.* We apply the theory of symmetric functions, specifically Schur functions (see Macdonald [17]). When $d_1, \ldots, d_t$ are integers with

$$1 \leqslant d_1 < d_2 < \cdots < d_t,$$

we define the polynomial $K(\mathbf{x}; \mathbf{d})$ by means of the relation

$$\det(x_j^{d_i - 1})_{1 \leqslant i,j \leqslant t} = K(\mathbf{x}; \mathbf{d})V_t(\mathbf{x}). \tag{4}$$

For the sake of concision, we make use of the notation used in Macdonald [17]. Thus, by equation (3.1) of [17, Chapter I], one has $K(\mathbf{x}; \mathbf{d}) = s_\lambda$, where $\lambda$ is the partition

$$(d_t - t, d_{t-1} - (t-1), \ldots, d_1 - 1).$$

But equation (5.12) of [17, Chapter I] shows that $s_\lambda = \sum_T x^T$, where the summation is over all semi-standard tableaux $T$ of shape $\lambda$, and here, if the weight of $T$ is $\alpha = (\alpha_1, \ldots, \alpha_t)$, then $x^T$ is the monomial $x_1^{\alpha_1} \cdots x_t^{\alpha_t}$. Note that if $\lambda = (0, \ldots, 0)$, then one adopts the convention that $s_\lambda = 1$.

Observe next that by elementary properties of determinants, the polynomial

$$\det(f_i'(x_j))_{1 \leqslant i,j \leqslant t} \tag{5}$$

is a linear combination of polynomials of the shape

$$\det(x_j^{d_i - 1})_{1 \leqslant i,j \leqslant t}, \tag{6}$$

with $1 \leqslant d_i \leqslant k_i$ $(1 \leqslant i \leqslant t)$. A moment's reflection here reveals that this linear combination contains the polynomial

$$\det(x_j^{k_i - 1})_{1 \leqslant i,j \leqslant t},$$

with a non-vanishing coefficient. By permuting rows within the determinants (6), there is no loss of generality in supposing that $1 \leqslant d_1 \leqslant d_2 \leqslant \ldots \leqslant d_t$. Then each such determinant contributes either 0, or else a polynomial of the shape (4), to the expansion of (5). We may therefore conclude that

$$\det(f_i'(x_j))_{1 \leqslant i,j \leqslant t} = \widehat{K}(\mathbf{x}; \mathbf{k})V_t(\mathbf{x}),$$

where $\widehat{K}(\mathbf{x};\mathbf{k})$ is a polynomial of total degree

$$d=\sum_{i=1}^{t}(k_i-1)-\sum_{i=1}^{t}(i-1),$$

in which the homogeneous part of highest degree is a non-zero multiple of $s_\Lambda$, with $\Lambda=(k_t-t,k_{t-1}-(t-1),\dots,k_1-1)$. But in view of the discussion concluding the previous paragraph, whenever $x_i>B>1$ $(1\leqslant i\leqslant t)$, one has $s_\Lambda\geqslant B^d$, and thus we conclude that $\widehat{K}(\mathbf{x};\mathbf{k})$ is asymptotically definite. This completes the proof of the lemma.

Define next the polynomials $\phi_{i,s}=\phi_{i,s}(\mathbf{x};\mathbf{f})$ by taking

$$\phi_{i,s}(\mathbf{x})=f_i(x_1)+\dots+f_i(x_s)\quad(1\leqslant i,s\leqslant t).$$

Our next lemma establishes the existence of an eliminant polynomial suitable for subsequent deliberations.

**Lemma 2.** *Suppose that $t\geqslant 2$, and that $(f_1,\dots,f_t)$ is a well-conditioned $t$-tuple of polynomials. Then there exists a polynomial $\Psi(\mathbf{z})\in\mathbb{Z}[z_1,\dots,z_t]$, with total degree and coefficients depending at most on $t$, $\mathbf{k}$ and the coefficients of $f_1,\dots,f_t$, such that*

$$\Psi(\phi_{1,t-1}(\mathbf{x}),\dots,\phi_{t,t-1}(\mathbf{x}))=0, \tag{7}$$

*and yet*

$$\Psi(\phi_{1,t}(\mathbf{x}),\dots,\phi_{t,t}(\mathbf{x}))\neq 0. \tag{8}$$

*Proof.* The existence of a non-trivial polynomial $\Psi(\mathbf{z})\in\mathbb{Z}[z_1,\dots,z_t]$, for which $\Psi(\phi_{1,t-1},\dots,\phi_{t,t-1})$ is identically zero, follows by considering transcendence degrees. Let $K=\mathbb{Q}(\phi_{1,t-1},\dots,\phi_{t,t-1})$. Then $K\subseteq\mathbb{Q}(x_1,\dots,x_{t-1})$, so that $K$ has transcendence degree at most $t-1$ over $\mathbb{Q}$. But then the $t$ polynomials $\phi_{i,t-1}(\mathbf{x})\in K$ $(1\leqslant i\leqslant t)$ cannot be algebraically independent, whence the existence of the above polynomial $\Psi$ follows immediately.

In order to verify the condition (8), consider any non-trivial polynomial $\Psi$ of smallest total degree for which the polynomial equation (7) holds, and suppose, if possible, that $\Psi(\phi_{1,t}(\mathbf{x}),\dots,\phi_{t,t}(\mathbf{x}))$ is identically zero. Then the polynomials

$$(\partial/\partial x_i)\Psi(\phi_{1,t}(\mathbf{x}),\dots,\phi_{t,t}(\mathbf{x}))\quad(1\leqslant i\leqslant t)$$

are also identically zero. On applying the chain rule, we therefore find that

$$\sum_{j=1}^{t}f_j'(x_i)\Psi_j(\phi_{1,t}(\mathbf{x}),\dots,\phi_{t,t}(\mathbf{x}))=0\quad(1\leqslant i\leqslant t), \tag{9}$$

where we have written $\Psi_j(\mathbf{z})$ for $(\partial/\partial z_j)\Psi(\mathbf{z})$. But as a consequence of Lemma 1, the polynomial

$$\det(f_j'(x_i))_{1\leqslant i,j\leqslant t}$$

is not identically zero, and thus it follows from (9) that each of the polynomials $\Psi_j(\phi_{1,t}(\mathbf{x}),\dots,\phi_{t,t}(\mathbf{x}))$ $(1\leqslant j\leqslant t)$ must be identically zero. However, since $\Psi(\mathbf{z})$ is a non-constant polynomial, at least one of the derivatives $\Psi_j(\mathbf{z})$ $(1\leqslant j\leqslant t)$ must be non-zero. Thus there exists a non-trivial polynomial $\Psi_j(\mathbf{z})\in\mathbb{Z}[\mathbf{z}]$ for which, in particular, one has

$$\Psi_j(\phi_{1,t-1}(\mathbf{z}),\dots,\phi_{t,t-1}(\mathbf{z}))=0.$$

Since the latter conclusion contradicts the minimality of the total degree of $\Psi$, we are forced to conclude that the inequality (8) does indeed hold.

**3. Application of the eliminant polynomial.** We are now equipped to prosecute our proof of Theorem 1. We begin by noting that the conclusion of the theorem is classical when $t = 1$. For on writing $e(z)$ for $\exp(2\pi iz)$, and

$$F(\alpha) = \sum_{1 \leqslant x \leqslant P} e(\alpha f_1(x)),$$

it follows from Hua's lemma (see, for example, Theorem 4 of Hua [16]) that whenever $k_1 \geqslant 2$ and $P \geqslant 3$, one has

$$S_2(P; f_1) = \int_0^1 |F(\alpha)|^4 d\alpha \ll P^2 (\log P)^A,$$

where $A$ is a positive number depending at most on $k_1$ and the coefficients of $f_1$. We may therefore suppose in what follows that $t \geqslant 2$, and moreover the discussion of §2 permits the assumption that $(f_1, \dots, f_t)$ is well-conditioned. We thus infer from Lemma 1 that there exists an asymptotically definite polynomial $\Theta(\mathbf{x}; \mathbf{f})$ with the property that the relation (3) holds. We write $C$ for the parameter associated with $\Theta(\mathbf{x}; \mathbf{f})$ from our definition of asymptotic definiteness.

We next dispose of small solutions of (1) counted by $S_{t+1}(P; \mathbf{f})$. Write

$$G(\boldsymbol{\alpha}) = \sum_{1 \leqslant x \leqslant C} e(\alpha_1 f_1(x) + \dots + \alpha_t f_t(x))$$

and

$$H(\boldsymbol{\alpha}) = \sum_{C < x \leqslant P} e(\alpha_1 f_1(x) + \dots + \alpha_t f_t(x)).$$

Then by orthogonality one finds that

$$\begin{aligned} S_{t+1}(P; \mathbf{f}) &= \int_{[0,1)^t} |G(\boldsymbol{\alpha}) + H(\boldsymbol{\alpha})|^{2t+2} d\boldsymbol{\alpha} \\ &\ll \int_{[0,1)^t} |G(\boldsymbol{\alpha})|^{2t+2} d\boldsymbol{\alpha} + \int_{[0,1)^t} |H(\boldsymbol{\alpha})|^{2t+2} d\boldsymbol{\alpha}. \end{aligned}$$

Thus, on writing $S_s^*(P; \mathbf{f})$ for the number of integral solutions of (1) with $C < x_i, y_i \leqslant P$ $(1 \leqslant i \leqslant s)$, a trivial estimate for $G(\boldsymbol{\alpha})$ yields the upper bound

$$S_{t+1}(P; \mathbf{f}) \ll 1 + S_{t+1}^*(P; \mathbf{f}).$$

Let $S_{t+1}^0(P; \mathbf{f})$ denote the number of integral solutions of the system

$$\sum_{i=1}^t f_j(x_i) - f_j(x_{t+1}) = \sum_{i=1}^t f_j(y_i) - f_j(y_{t+1}) \quad (1 \leqslant j \leqslant t), \tag{10}$$

with $C < x_i, y_i \leqslant P$ $(1 \leqslant i \leqslant t)$, satisfying the condition that $x_i = x_j$ for some $1 \leqslant i < j \leqslant t$. Also, let $\widehat{S}_{t+1}(P; \mathbf{f})$ denote the complementary number of solutions of (10) in which $x_i \neq x_j$ for $1 \leqslant i < j \leqslant t$. Then plainly

$$S_{t+1}^*(P; \mathbf{f}) = S_{t+1}^0(P; \mathbf{f}) + \widehat{S}_{t+1}(P; \mathbf{f}). \tag{11}$$

But it follows from a consideration of the underlying diophantine equations that

$$\begin{aligned} S_{t+1}^0(P; \mathbf{f}) &\ll \int_{[0,1)^t} H(2\boldsymbol{\alpha}) H(\boldsymbol{\alpha})^{t-1} H(-\boldsymbol{\alpha})^{t+1} d\boldsymbol{\alpha} \\ &\ll \int_{[0,1)^t} |H(2\boldsymbol{\alpha}) H(\boldsymbol{\alpha})^{2t}| \, d\boldsymbol{\alpha}, \end{aligned}$$

and thus, by Hölder's inequality, we find that

$$S_{t+1}^0(P; \mathbf{f}) \ll \Big( \int_{[0,1)^t} |H(\boldsymbol{\alpha})|^{2t+2} d\boldsymbol{\alpha} \Big)^{t/(t+1)} \Big( \int_{[0,1)^t} |H(2\boldsymbol{\alpha})|^{2t+2} d\boldsymbol{\alpha} \Big)^{1/(2t+2)}.$$

Then on considering the underlying diophantine equations, we deduce that

$$S^0_{t+1}(P;\mathbf{f}) \ll \left(S^*_{t+1}(P;\mathbf{f})\right)^{(2t+1)/(2t+2)},$$

whence by (2) and (11),

$$S^*_{t+1}(P;\mathbf{f}) \ll \widehat{S}_{t+1}(P;\mathbf{f}). \tag{12}$$

We now analyse the solutions of (10) counted by $\widehat{S}_{t+1}(P;\mathbf{f})$. By Lemma 2, there exists a polynomial $\Psi(\mathbf{z}) \in \mathbb{Z}[z_1,\dots,z_t]$, with total degree and coefficients depending at most on $t$, $\mathbf{k}$ and the coefficients of $f_1,\dots,f_t$, such that whenever $u_i = u_{t+1}$ for some $1 \leqslant i \leqslant t$, one has

$$\Psi(\phi_{1,t}(\mathbf{u}) - f_1(u_{t+1}),\dots,\phi_{t,t}(\mathbf{u}) - f_t(u_{t+1})) = 0,$$

and yet

$$\Psi(\phi_{1,t}(\mathbf{u}),\dots,\phi_{t,t}(\mathbf{u})) \neq 0.$$

It follows that for some non-trivial polynomial $\Phi(\mathbf{u}) \in \mathbb{Z}[u_1,\dots,u_{t+1}]$, one has

$$\Psi(\phi_{1,t}(\mathbf{u}) - f_1(u_{t+1}),\dots,\phi_{t,t}(\mathbf{u}) - f_t(u_{t+1})) = \Phi(\mathbf{u})\prod_{i=1}^{t}(u_i - u_{t+1}). \tag{13}$$

For the sake of concision, we write

$$\Upsilon(\mathbf{z}) = \Phi(\mathbf{z})\prod_{i=1}^{t}(z_i - z_{t+1}).$$

Let $T_1$ denote the number of solutions of (10) counted by $\widehat{S}_{t+1}(P;\mathbf{f})$ having the property that $\Upsilon(\mathbf{y})$ is non-zero, and let $T_2$ denote the corresponding number of solutions with $\Upsilon(\mathbf{y}) = 0$. Then

$$\widehat{S}_{t+1}(P;\mathbf{f}) = T_1 + T_2. \tag{14}$$

Consider first a solution $(\mathbf{x},\mathbf{y})$ counted by $T_1$. In view of (10) and (13), one has

$$\Phi(\mathbf{x})\prod_{i=1}^{t}(x_i - x_{t+1}) = \Phi(\mathbf{y})\prod_{i=1}^{t}(y_i - y_{t+1}). \tag{15}$$

Fix a choice of $\mathbf{y}$ with $\Upsilon(\mathbf{y}) \neq 0$. Then if $\tau(n)$ denotes the divisor function, we find from (15) that there are at most $(2\tau(|\Upsilon(\mathbf{y})|))^t$ possible choices for $x_i - x_{t+1}$ $(1 \leqslant i \leqslant t)$. Fixing any one such choice of the latter $t$ quantities, we write $x_i = x_{t+1} + d_i$ $(1 \leqslant i \leqslant t)$. Then on substituting these fixed choices of $\mathbf{y}$ and $\mathbf{d}$ into (10), we find that $x_{t+1}$ satisfies the evidently non-trivial equation

$$\sum_{i=1}^{t} f_1(x_{t+1} + d_i) - f_1(x_{t+1}) = \sum_{i=1}^{t} f_1(y_i) - f_1(y_{t+1}).$$

One therefore has $O(1)$ possibilities for $x_{t+1}$, whence the total number of solutions of this type is

$$T_1 \ll \sum_{\mathbf{y}} \left(\tau(|\Upsilon(\mathbf{y})|)\right)^t,$$

where the summation is over $\mathbf{y}$ with $1 \leqslant y_i \leqslant P$ $(1 \leqslant i \leqslant t+1)$ and $\Upsilon(\mathbf{y}) \neq 0$. We thus conclude from Theorem 3 of Hua [16] that

$$T_1 \ll P^{t+1}(\log P)^A, \tag{16}$$

where the positive number $A$ depends at most on $t$, $\mathbf{k}$ and the coefficients of $f_1,\dots,f_t$.

Next consider a solution $(\mathbf{x},\mathbf{y})$ counted by $T_2$. The number of values of $\mathbf{y}$ with $1 \leqslant y_i \leqslant P$ $(1 \leqslant i \leqslant t+1)$ for which $\Upsilon(\mathbf{y}) = 0$ is $O(P^t)$ (see, for example, the proof of Lemma 2 of Wooley [28]). Fix any one such choice of $\mathbf{y}$, and any one of the $O(P)$ possible choices for $x_{t+1}$. Then on writing

$$N_j = f_j(x_{t+1}) - f_j(y_{t+1}) + \sum_{i=1}^{t} f_j(y_i) \quad (1 \leqslant j \leqslant t),$$

we find from (10) that

$$\sum_{i=1}^{t} f_j(x_i) = N_j \quad (1 \leqslant j \leqslant t). \tag{17}$$

Suppose first that $\mathbf{x}$ is a singular solution of (17). Then one has

$$\det(f_i'(x_j))_{1\leqslant i,j\leqslant t} = 0,$$

whence from (3) we have

$$\Theta(\mathbf{x};\mathbf{f}) \prod_{1\leqslant i<j\leqslant t} (x_j - x_i) = 0. \tag{18}$$

But by hypothesis, one has $x_i \neq x_j$ for $1 \leqslant i < j \leqslant t$, and moreover Lemma 1 ensures that since $x_i > C$ $(1 \leqslant i \leqslant t)$, one has $\Theta(\mathbf{x};\mathbf{f}) \neq 0$. Then the equation (18) is impossible, whence there are no singular solutions $\mathbf{x}$ counted by $T_2$.

We complete our treatment of $T_2$ by considering the non-singular points $\mathbf{x}$ satisfying (17). According to Theorem 7.7 of Hartshorne [5, Chapter 1], the number of irreducible components contained in the intersection (17) is at most $k_1k_2\cdots k_t$. If such a component has positive dimension, then it arises from an improper intersection and is consequently singular. Thus it follows that all the points that concern us here arise from components of the intersection having dimension zero, whence their number is also at most $k_1k_2\cdots k_t$. Then we may conclude that for the fixed choice of $(x_{t+1},\mathbf{y})$ under consideration, there are $O(1)$ permissible choices of $x_1,\ldots,x_t$. Finally, therefore, we deduce that

$$T_2 \ll P^{t+1}. \tag{19}$$

On combining (12), (14), (16) and (19), we at last arrive at the upper bound

$$S_{t+1}(P;\mathbf{f}) \ll P^{t+1}(\log P)^A,$$

and this completes the proof of our theorem.

## References


1. M. A. Bennett, N. P. Dummigan and T. D. Wooley, *The representation of integers by binary additive forms*, Compositio Math. **111** (1998), 15–33.
2. G. Greaves, *On the representation of a number as a sum of two fourth powers*, Math. Z. **94** (1966), 223–234.
3. G. Greaves, *On the representation of a number as the sum of two fourth powers, II*, Mat. Zametki **55** (1994), 47–58.
4. G. Greaves, *Some diophantine equations with almost all solutions trivial*, Mathematika **44** (1997), 14–36.
5. R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, Berlin, 1977.
6. D. R. Heath-Brown, *The density of rational points on cubic surfaces*, Acta Arith. **79** (1997), 17–30.
7. D. R. Heath-Brown, *The density of rational points on curves and surfaces*, Ann. of Math. (2) **155** (2002), 553–595.
8. C. Hooley, *On the representations of a number as the sum of two cubes*, Math. Z. **82** (1963), 259–266.
9. C. Hooley, *On the representation of a number as a sum of two hth powers*, Math. Z. **84** (1964), 126–136.
10. C. Hooley, *On binary cubic forms*, J. Reine Angew. Math. **226** (1967), 30–87.
11. C. Hooley, *On the numbers that are representable as the sum of two cubes*, J. Reine Angew. Math. **314** (1980), 146–173.
12. C. Hooley, *On another sieve method and the numbers that are a sum of two hth powers*, Proc. London Math. Soc. (3) **43** (1981), 73–109.
13. C. Hooley, *On binary quartic forms*, J. Reine Angew. Math. **366** (1986), 32–52.
14. C. Hooley, *On another sieve method and the numbers that are a sum of two hth powers: II*, J. Reine Angew. Math. **475** (1996), 55–75.
15. C. Hooley, *On binary cubic forms. II*, J. Reine Angew. Math. **521** (2000), 185–240.
16. L.-K. Hua, *Additive theory of prime numbers*, Amer. Math. Soc., Providence, Rhode Island, 1965.
17. I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford Mathematical Monographs, Oxford, 1979.
18. S. T. Parsell, *The density of rational lines on cubic hypersurfaces*, Trans. Amer. Math. Soc. **352** (2000), 5045–5062.
19. S. T. Parsell, *Pairs of additive equations of small degree*, Acta Arith. **104** (2002), 345–402.
20. C. M. Skinner and T. D. Wooley, *Sums of two kth powers*, J. Reine Angew. Math. **462** (1995), 57–68.
21. C. M. Skinner and T. D. Wooley, *On the paucity of non-diagonal solutions in certain diagonal Diophantine systems*, Quart. J. Math. Oxford (2) **48** (1997), 255–277.
22. W. Y. Tsui and T. D. Wooley, *The paucity problem for simultaneous quadratic and biquadratic equations*, Math. Proc. Cambridge Philos. Soc. **126** (1999), 209–221.
23. R. C. Vaughan, *A new iterative method in Waring's problem*, Acta Math. **162** (1989), 1–71.
24. R. C. Vaughan and T. D. Wooley, *Further improvements in Waring's problem*, Acta Math. **174** (1995), 147–240.
25. R. C. Vaughan and T. D. Wooley, *On a certain nonary cubic form and related equations*, Duke Math. J. **80** (1995), 669–735.

26. R. C. Vaughan and T. D. Wooley, *A special case of Vinogradov's mean value theorem*, Acta Arith. **79** (1997), 193–204.
27. R. C. Vaughan and T. D. Wooley, *Further improvements in Waring's problem, IV: higher powers*, Acta Arith. **94** (2000), 203–285.
28. T. D. Wooley, *A note on symmetric diagonal equations*, Number Theory with an emphasis on the Markoff spectrum (Provo, UT, 1991) (A. D. Pollington and W. Moran, eds.), Dekker, New York, 1993, pp. 317–321.
29. T. D. Wooley, *Sums of two cubes*, Internat. Math. Res. Notices (1995), 181–185.
30. T. D. Wooley, *An affine slicing approach to certain paucity problems*, Analytic Number Theory: Proceedings of a Conference in Honor of Heini Halberstam (B. C. Berndt, H. G. Diamond, and A. J. Hildebrand, eds.), vol. 2, 1996, pp. 803–815.
31. T. D. Wooley, *On exponential sums over smooth numbers*, J. Reine Angew. Math. **488** (1997), 79–140.
32. T. D. Wooley, *Sums and differences of two cubic polynomials*, Monatsh. Math. **129** (2000), 159–169.



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